Filtering in Fractional Fourier Domains: Application to the Biot's Waves

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Abstract

In this paper we present a method to separate the compressional waves which propagate in a porous medium when it is subject to a mechanical excitation. We start this work by reviewing the Biot's theory which describes the propagation of ultrasonic pulses in a porous elastic medium. This modelling shows that three kinds of waves propagate in such media: two compressional waves and one shear wave, each one with its own velocity. Because of the dispersive nature of porous media, the identification of the compressional waves is often difficult by a traditionnal filtering while this identification is a compelling need to extract the part of the informations about the elastic parameters, the porosity and the permeability of the medium contained in each of them. For that we propose a filtering method using the fractional Fourier transform as foundation. The interpretation of this transformation as a rotation in the time-frequency plane and its relationships with time-frequency representations allow the filtering of signal in a single fractional Fourier domain.

1 Introduction

The fundamental mechanical characteristics of any material are the density and the bulk moduli. Bulk moduli are scalar constants which relate the stress field in a medium when it is subjected to strains, and for a large majority of media, they are the constants related to the phase velocities as well as the attenuation of the waves as they propagate through the medium. When the medium under consideration is composed of several components, the mechanical characteristics depend on the characteristics of each component and of the geometrical and coupling constants which describe the relative motion of the components and their exchanges.
The porous media are a particular case of complex media and the propagation of waves through fluid saturated media is of great interest to scientists from several different disciplines. Studies involving wave propagation in fluid saturated media are common in medical imaging, in physics to understand the elastic properties of porous materials by testing their dynamical responses or in geophysics to estimate the porosity and fluid saturation of reservoir in petroleum research.
The Biot's theory provides an accurate description of the wave propagation in porous media. In particular, it predicts two compressional waves which exibit very different behaviours in nature, and so the richness of this response to mechanical excitations gives some hopes to extract the desired informations about the medium directly from the waveforms.
In the past, most of the parameters of porous materials have been determinated by non acoustical methods. However, acoustic waves and more generally propagation of acoustic pulses may provide a more realistic alternative source of information about the propagating medium. Unfortunately, because of the different couplings between the fluid and solid phases, these media are strongly dispersive in nature. This means that the wavepackets are spreading as they move through the media and that it is often difficult to separate the propagating modes by classical filtering methods. However time-frequency representation (TFR) of the received waveforms shows that the TFR of compressional waves may do not overlap and that the separation of compressional waves is then possible in the time-frequency plane. Our approach to do that is based on the Fractional Fourier Transform (FRFT) which generalizes the filtering in the single time domain or frequency domain to filtering in a single fractional Fourier domain.
The outline of this paper is as follows: in section 2, the Biot's model is presented, section 3 is devoted to the Fractional Fourier Transform and the application of the filtering of Biot's waves is given in section 4.

2 The Biot's theory

The Biot's theory of elastic waves propagation is an effective medium theory of porous media saturated by a fluid [1]. This must be understood in the following sens: the Biot's theory considers a nonhomogeneous material consisting of a solid phase matrix with interconnected void space filled by a fluid (gas or liquid) as a continuum, the physical parameters of which are determined as functions of those of each constituant of the material and it

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models both individual and coupled behaviour of the solid and pore fluid. The main succes of that modelling is the prediction of two compressional and one shear waves. One of the compressional waves propagates faster than the other and is termed the "fast wave" or the compressional wave of first order. The slower one is the "Biot's slow wave" or the compressional wave of second order. To take into account the viscous and thermal exchanges between the frame and the pore fluid, Johnson [2] and Allard and Lafarge [3,4] introduced two correction factors: the dynamical tortuosity $\alpha(\omega)$ and dynamical compressibility $\beta(\omega)$ which correct respectively the density and the bulk modulus of the pore fluid. In the framework of the Biot's theory and for the high frequencies, the wave equations of each modes propagating in a fluid saturated porous medium have the following form [5]:

$$\frac{\partial^2 \Phi_i}{\partial x^2} - a_i \frac{\partial^2 \Phi_i}{\partial t^2} + b_i \frac{\partial^{3/2} \Phi_i}{\partial t^{3/2}} + c_i \frac{\partial \Phi_i}{\partial t} = 0. \tag{1}$$

$\Phi_i(x,t)$ are the fields of the fast (i=+), slow (i=-) and shear (i=sh) waves and a_i, b_i and c_i are the coefficients of their respective equations related to the velocity, the attenuation and the dispersion of the corresponding wave. Their values are obtained by tedious calculus and we give here only the coefficients corresponding to their velocities:

$$\frac{1}{v_+^2} = a_+ = \frac{\rho_0 \alpha_\infty}{K_f}, \quad \frac{1}{v_-^2} = a_- = \frac{(1-\phi)\rho_s}{K_b + \frac{4}{3}N} \quad \text{and} \quad \frac{1}{v_{sh}^2} = a_{sh} = \frac{(1-\phi)\rho_s + \phi\rho_0(1-1/\alpha_\infty)}{N}. \tag{2}$$

where ϕ is the porosity of the medium, ρ_o and K_f are respectively the density and the bulk modulus of the fluid ρ_s, K_b and N are the density and the elastic constants of the bare skeletal frame, and α_∞ is the tortuosity of the medium.
At low frequency, the dynamics of the fluid is dominated by the viscous effects. The pore fluid is locked with the frame such that there is no relative motion between the solid frame and the fluid; the slow wave does not propagate. At high frequency, the viscous effects being localized near the surface of the pore, the inertial effects become predominant and cause the motion of the fluid to be almost independant of that of the frame. This behaviour results in velocity dispersion and frequency dependant attenuation which are influenced by the geometry of the pore and the viscosity of the fluid. The nature and the characteristics of the waves propagating in a fluid saturated medium are quite different and are strongly dependant of the properties of the components of the medium. The fast and the shear wave are weakly damped while the slow wave exibits a high attenuation over a wide range of frequencies and hence is difficult to detect. In the same way, it can be understood that each wave has its own velocity dispersion. These phenomena lead to quite different pictures in the time-frequency plane for each of the compressional waves and, from this fact, one can reasonably hope that their identification and their separation are possible.

3 Fractional Fourier Transform

The FRFT was introduced by Namias [6] in the framework of quantum mechanics where it provides an efficient tool to solve some classes of differential equations. Since this work several applications of the FRFT have been suggested and developped [7,8].

3.1 Definition and properties

The α-th order FRFT of a function denoted by $\{\mathcal{F}^\alpha x\}(t_\alpha)$ is defined for $\alpha \in [0,4]$ by

$$\{\mathcal{F}^\alpha x\}(t_\alpha) = x_\alpha(t_\alpha = \int_{-\infty}^{\infty} \mathcal{K}_\alpha(t_\alpha, t')x(t')dt' \quad \text{where} \quad \mathcal{K}_\alpha(t_\alpha, t') = A_\phi \exp\left[i\pi((t_\alpha^2 \cot\phi - 2t_\alpha t' \csc\phi + t'^2 \cot\phi)\right] \tag{3}$$

with $A_\phi = \exp\left[-i\pi sgn(\sin\phi)/4 + i\phi/2\right]/\sqrt{|sin\phi|}$ and $\phi = \alpha\pi/2$. The most important properties of FRFT are the following: 1) it is linear, 2) the first order transform ($\alpha = 1$) corresponds to the common Fourier transform, 3) it is an unitary transform. Other properties can be deduced from those of the kernel: $\mathcal{K}_\alpha(t,u) = \mathcal{K}_\alpha(u,t)$, $\mathcal{K}_{-\alpha}(t,u) = \mathcal{K}_\alpha^*(t,u)$, $\mathcal{K}_\alpha(-t,u) = \mathcal{K}_\alpha(t,-u)$ where the superscript * represents the complex conjugation operation. The kernel functions taken as functions of t with parameter u belong to an orthonormal set and form a one parameter group:

$$\int_{-\infty}^{\infty} \mathcal{K}_\alpha(t,u)\mathcal{K}_\alpha^*(t,u')dt = \delta(u-u'), \quad \text{and} \quad \int_{-\infty}^{\infty} \mathcal{K}_\alpha(t,t')\mathcal{K}_\beta(t',u)dt' = \mathcal{K}_{\alpha+\beta}(t,u). \tag{4}$$

The inverse transform can be derived from these properties as $x(t) = \int \mathcal{K}_{-\alpha}(t,t_\alpha)[\{\mathcal{F}^\alpha x\}(t_\alpha)]dt_\alpha$.

3.2 Relationships with the time-frequency distributions

The group property of the $\mathcal{K}_\alpha(t,u)$ functions and the particular cases $\alpha = 0$ and $\alpha = 1$ result in the interpretation of the FRFT as the signal distribution along a fractional domain t_α between time and frequency. In fact the

FRFT is related to the Wigner distribution. As it is well known, the Wigner distribution $W_x(t,f)$ of the function $x(t)$ defined by:

$$W_x(t,f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau} x\big(t+\frac{\tau}{2}\big)x^*\big(t-\frac{\tau}{2}\big)d\tau \qquad (5)$$

is the time-frequency energy distribution of the signal and its projection onto the t (resp.f) axis gives the magnitude squared of the time (resp. frequency) domain representation:

$$\int_{-\infty}^{\infty} W_x(t,f)df = |x(t)|^2 = \int_{-\infty}^{\infty} W_x(t,f)dt = |x_1(f)|^2, \qquad (6)$$

where $x_1(f) = x_1(t_1)$ is the $x(t)$ Fourier transform. The generalization of this property leads to the following relation

$$\{\mathcal{R}_\phi[W_x(t,f)]\} = |x_\alpha(t_\alpha)|^2, \qquad (7)$$

where $\mathcal{R}_\phi$ is the Radon transform operator which takes the integral projection of the fonction $W_x(t,f)$ onto an axis making angle ϕ with the t axis.

3.3 Filtering in Fractional Fourier domains

Let us consider a signal corrupted by an additive noise such that their Fourier transforms do not overlap. Then it is easy to eliminate the undesirable frequencies by a suitable filter in the frequency domain. It is also easy to recover a signal corrupted by a noise if the supports of their time representations have an empty intersection. When the signal and noise Wigner distributions do not overlap and if their projections on both the t and f axes overlap, then one cannot exactly remove the noise effects by filtering in time or frequency domain. On the other hand, it may be possible to find values of ϕ for which the the Radon transforms of the time-frequency pictures do not intersect (Fig.1). So in these fractional Fourier domains, one can exactly eliminate the noise term from the signal [9].

4 Application

As an application of the method of the fractional Fourier domain filtering we wish to separate the compressional waves within the response of a slab of porous material saturated by water submitted to an incident ultrasonic pulse.
The incident pulse produced by an ultrasonic transducer is a monochromatic wave modulated by a gaussian function. The input (incident) and received (transmitted) signals and their TFR are shown in Fig.2. The waveforms of the compressional waves are quite different: the amplitude of the slow wave is more reduced, and apparently its shape is extended with smooth variations. This is confirmed by the TFR: the spectral content of both the incident and fast waves are very similar while the one of the slow wave shows a more important velocity dispersion and a stronger attenuation in high frequencies range.
The FRFT of the input signal is plotted on Fig.3-a for $\alpha \in [0,1]$. As α goes from 0 to 1, the FRFT splits off in two identical parts which more and more look like to gaussian functions symmetrically shifted along the the t_α axis.
An exemple of the reconstucted signal by inverse FRFT in the fractional Fourier domain $\alpha = .7$ is given on Fig.3-b. Only one part of the FRFT (solid line) is used to reconstruct the signal. Each part of the FRFT having the same spectral content contribues to a half of the reconstruction of the time representation of the signal. Fig.4-a shows the modulus of the FRFT of Biot's waves for $\alpha \in [0,1]$. As the FRFT of each wave has two components, the FRFT of the transmitted signal has a quite complex structure. It is the sum of three contributions: a central part which is the Radon tranform of the crossterms of the Wigner distribution and an other part on each side (Fig.4-b). For the small values of α, ($\alpha < 0.3$), these three contributions partely overlap: the waves cannot be filtred. For $0.4 < \alpha < 0.75$, the central part is more and more important but in this range, the three parts are coming apart. In the corresponding fractional Fourier domains the Biot's waves can be separated by simmple filtering. For higher values of the order, the FRFT of the transmitted signal is close to the Fourier transform and the filtering is difficult if not impossible. The filtering in the fractional Fourier domain $\alpha = 0.62$ is illustrated on Fig.5. On the upper subplots are the FRFT; the solide lines show the part of the FRFT used to recover the Biot's waves fast wave on the left and slow wave on the right . The lower subplots represent the filtred Biot's waves. As noticed above, each side part of the FRFT only contribues to a half of the amplitude of the signal, the lost information is contained in the central part of the FRFT which is rejected by the filtering process.

5 Conclusion

In this paper we have descibed a method to analyse and to separate the Biot's waves which propagate in fluid saturated porous materials. The results obtained are encouraging and show that the FRFT is an effecient tool, but they are still too dependant of the knowledge about the incident signal. Several remaing problems must be considered as for exemple real-time implementation of the method, noise effects to improve the method.

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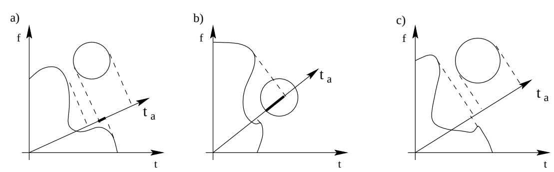

Figure 1: Filtering in fractional Fourier domain.

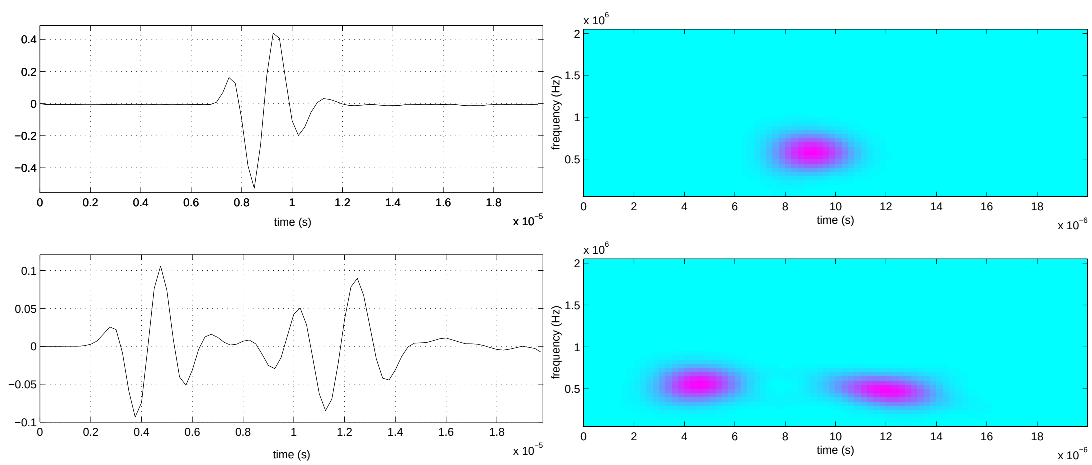

Figure 2: Incident and transmitted signals (a) and their TF representations (b).

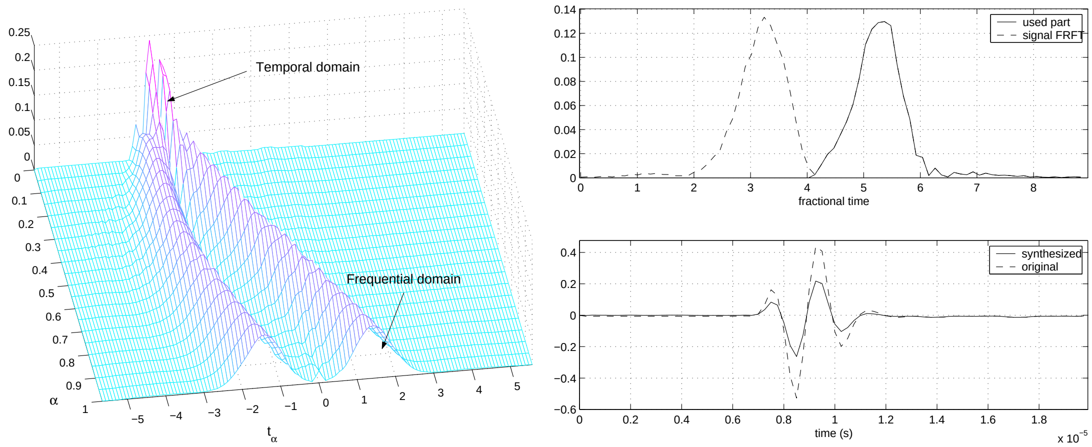

Figure 3: Modulus of the FRFT of the incident signal for $\alpha \in [0,1]$(a); reconstruction of the signal by fractional filtering and inverse FRFT (b).

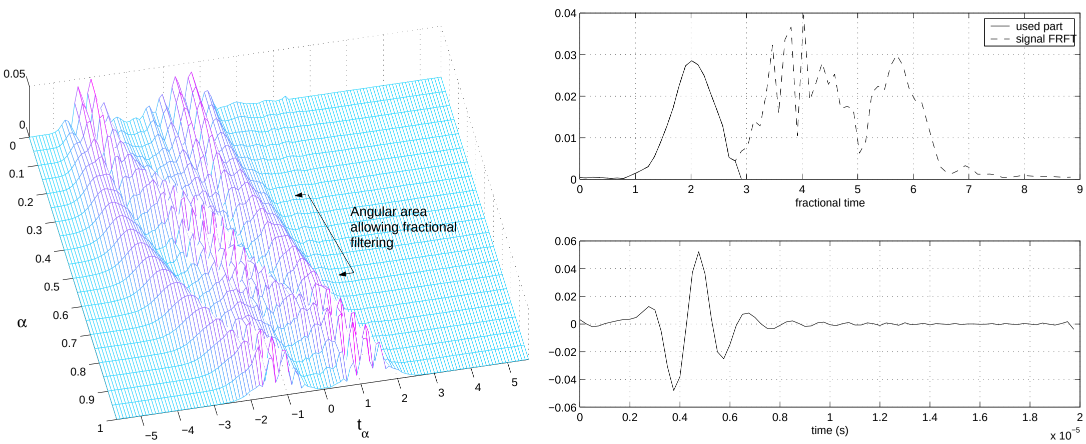

Figure 4: Modulus of the FRFT of the transmitted signal for $\alpha \in [0,1]$ (a); FRFT of the same signal for $\alpha = 0.55$ (b).

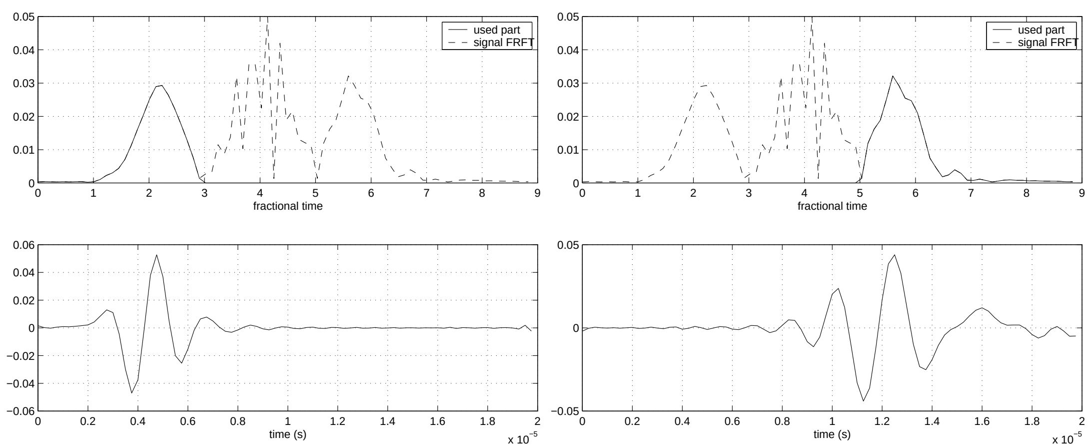

Figure 5: Filtering of the Biot's waves in the $\alpha = 0.62$ fractional Fourier domain: fast wave on the left and slow wave on the right.